

A family of new simple modules over the Schrödinger-Virasoro algebra *

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Abstract: In this article, a large class of simple modules over the Schrödinger-Virasoro algebra \mathcal{G} are constructed, which include highest weight modules and Whittaker modules. These modules are determined by the simple modules over the finite-dimensional quotient algebras of some subalgebras. Moreover, we show that all simple modules of \mathcal{G} with locally finite actions of elements in a certain positive part belong to this class of simple modules. Similarly, a large class of simple modules over the W -algebra $W(2, 2)$ are constructed.

Key words: Schrödinger-Virasoro algebra, W -algebra, Highest weight module, Whittaker module, Simple module.

Mathematics Subject Classification (2010): 17B10, 17B20, 17B65, 17B66, 17B68.

1 Introduction

Throughout this paper, we denote by $\mathbb{C}, \mathbb{Z}, \mathbb{N}$ and \mathbb{Z}_+ the sets of complex numbers, integers, nonnegative integers and positive integers, respectively. All vector spaces and Lie algebras are over \mathbb{C} . For a Lie algebra \mathcal{L} , we denote by $\mathcal{U}(\mathcal{L})$ the universal enveloping algebra of \mathcal{L} .

The Schrödinger-Virasoro algebra is an extension of the Virasoro Lie algebra by a nilpotent Lie algebra formed with a bosonic current of weight $\frac{3}{2}$ and a bosonic current of weight 1. It was introduced in the context of non-equilibrium statistical physics during the process of investigating the free Schrödinger equations (see [6]). From then on, the Schrödinger-Virasoro algebra attracted a lot of attentions from researchers (see, e.g., [7, 11, 18–20, 23]). Now we recall the definition of the *Schrödinger-Virasoro algebra* \mathcal{G} , which is an infinite dimensional Lie algebra with the \mathbb{C} -basis $\{M_m, Y_{m+\frac{1}{2}}, L_m, C \mid m \in \mathbb{Z}\}$ and the following Lie brackets:

$$\begin{aligned} [L_m, L_n] &= (n - m)L_{m+n} + \delta_{m+n,0} \frac{m^3 - m}{12} C, \\ [L_m, Y_{n+\frac{1}{2}}] &= \left(n + \frac{1-m}{2}\right) Y_{m+n+\frac{1}{2}}, \quad [Y_{m+\frac{1}{2}}, Y_{n+\frac{1}{2}}] = (n - m)M_{m+n+1}, \\ [L_m, M_n] &= nM_{m+n}, \quad [M_m, M_n] = [M_m, Y_{n+\frac{1}{2}}] = [\mathcal{G}, C] = 0, \quad \forall m, n \in \mathbb{Z}. \end{aligned} \tag{1.1}$$

Note that the center of \mathcal{G} is spanned by $\{M_0, C\}$. In addition, the Schrödinger-Virasoro algebra is a special case for the generalized Schrödinger-Virasoro algebra (see [19]).

* Supported by the National Natural Science Foundation of China (No. 11371278, 11431010, 11501515) and the Zhejiang Provincial Natural Science Foundation of China (No. LQ16A010011).

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The highest weight modules and Whittaker modules are two classes of important modules, especially for infinite-dimensional Lie algebras such as Virasoro algebras, Heisenberg-Virasoro algebras, affine Kac-Moody algebras and so on. The construction of highest weight modules is one of the efficient ways to obtain simple weight modules (see, e.g., [3, 5, 9, 10, 15, 16]), while Whittaker modules become popular in recent years. It is well-known that Whittaker modules for $sl(2)$ were first discovered by Arnal and Pinczon in [2]. At the same time, the versions of Whittaker modules for finite dimensional complex semisimple Lie algebras were introduced by Kostant (see [8]). Since then, Whittaker modules over various Lie algebras draw a lot of attentions from the mathematicians and physicists (see, e.g., [1, 4, 12–14, 17]). Moreover, the Verma modules and Whittaker modules for the Schrödinger-Virasoro algebra are investigated in [19] and [24], respectively.

The actions of elements in the positive part of the algebra are locally finite, which is the same property of highest weight modules and Whittaker modules. This makes us study such a class of modules in a uniform way. Motivated by [3, 16], we construct a large family of new simple modules over the Schrödinger-Virasoro algebra. The highest weight modules and Whittaker modules are included and other modules, which are not weight modules, are new. Moreover, a class of new simple modules of $W(2, 2)$ are constructed similarly.

Let us now briefly describe how this paper is organized. In Section 2, we recall some fundamental definitions about what we need in the following. In Section 3, a class of new \mathcal{G} -modules are constructed, which are induced from simple modules over the finite-dimensional quotient algebras of some subalgebras. This result recovers one of the main results of Verma modules for a special case in [19] and the main results of Whittaker modules in [24]. In addition, it is shown that any simple module with locally finite actions of $M_i, (1 - \delta_{i,0})Y_{i-\frac{1}{2}}, L_i$ (or equivalently, locally nilpotent actions of $M_i, (1 - \delta_{i,0})Y_{i-\frac{1}{2}}, L_i$ as we shall prove) for sufficiently large $i \in \mathbb{N}$ must be one of the modules constructed above. In Section 4, some examples of simple \mathcal{G} -modules are presented. Finally, by the similar method, we describe new simple modules over the W -algebra $W(2, 2)$.

The main results of this paper are summarized in Theorems 3.1, 3.4, 5.1 and 5.2.

2 Preliminaries

In this section, we shall construct a class of induced \mathcal{G} -modules $\text{Ind}(V)$, where V is a simple module. First, some definitions and results for later use are recalled.

Definition 2.1. Let V be a module for a Lie algebra \mathbb{L} and $x \in \mathbb{L}$.

(1) If for any $v \in V$ there exists $n \in \mathbb{Z}_+$ such that $x^n v = 0$, then we call that the action of x on V is *locally nilpotent*. Similarly, the action of \mathbb{L} on V is *locally nilpotent* if for any $v \in V$ there exists $n \in \mathbb{Z}_+$ such that $\mathbb{L}^n x = 0$.

(2) If for any $v \in V$ we have $\dim(\sum_{n \in \mathbb{Z}_+} \mathbb{C} x^n v) < +\infty$, then we call that the action of

x on V is *locally finite*. Similarly, the action of \mathbb{L} on V is *locally finite* if for any $v \in V$ we have $\dim(\sum_{n \in \mathbb{Z}_+} \mathbb{L}^n v) < +\infty$.

It is easy to see that the action of x on V is locally nilpotent implies that the action of x on V is locally finite. If \mathbb{L} is a finitely generated Lie algebra, then we have that the action of \mathbb{L} on V is locally nilpotent implies that the action of \mathbb{L} on V is locally finite.

Denote by \mathbb{M} the set of all infinite vectors of the form $\underline{i} := (\dots, i_2, i_1)$ with entries in \mathbb{N} , satisfying the condition that the number of nonzero entries is finite. Let $\underline{0}$ denote the element $(\dots, 0, 0) \in \mathbb{M}$ and for $i \in \mathbb{Z}_+$ let ϵ_i denote the element $(\dots, 0, 1, 0, \dots, 0) \in \mathbb{M}$, where 1 is in the i 'th position from right. For any $\underline{i} \in \mathbb{M}$, we write

$$\mathbf{w}(\underline{i}) = \sum_{s \in \mathbb{Z}_+} s \cdot i_s,$$

which is a nonnegative integer. For any nonzero $\underline{i} \in \mathbb{M}$, let p and q be the largest and smallest integers such that $i_p \neq 0$ and $i_q \neq 0$ respectively, and define $\underline{i}' = \underline{i} - \epsilon_p$ and $\underline{i}'' = \underline{i} - \epsilon_q$.

Definition 2.2. (1) Denote by $>$ the *lexicographical total order* on \mathbb{M} , defined as follows: for any $\underline{i}, \underline{j} \in \mathbb{M}$

$$\underline{j} > \underline{i} \Leftrightarrow \text{there exists } r \in \mathbb{Z}_+ \text{ such that } (j_s = i_s, \forall s > r) \text{ and } j_r > i_r.$$

(2) Denote by \succ the *reverse lexicographical total order* on \mathbb{M} , defined as follows: for any $\underline{i}, \underline{j} \in \mathbb{M}$

$$\underline{j} \succ \underline{i} \Leftrightarrow \text{there exists } r \in \mathbb{Z}_+ \text{ such that } (j_s = i_s, \forall 1 \leq s < r) \text{ and } j_r > i_r.$$

Now we can induce a *principal total order* on $\mathbb{M} \times \mathbb{M} \times \mathbb{M}$, still denoted by \succ :

$$\begin{aligned} (\underline{i}, \underline{j}, \underline{k}) \succ (\underline{l}, \underline{m}, \underline{n}) &\Leftrightarrow (\underline{k}, \mathbf{w}(\underline{k})) \succ (\underline{n}, \mathbf{w}(\underline{n})) \quad \text{or} \\ \underline{k} = \underline{n} \text{ and } (\underline{j}, \mathbf{w}(\underline{j})) &\succ (\underline{m}, \mathbf{w}(\underline{m})) \quad \text{or} \\ \underline{k} = \underline{n}, \underline{j} = \underline{m} \text{ and } \underline{i} &> \underline{l}, \quad \forall \underline{i}, \underline{j}, \underline{k}, \underline{l}, \underline{m}, \underline{n} \in \mathbb{M}. \end{aligned}$$

For any $d_1, d_2 \in \mathbb{N}$ with $d_1 \geq 2d_2 - 1$, set

$$\mathcal{G}_{d_1, d_2} = \sum_{i \in \mathbb{N}} (\mathbb{C}M_{i-d_1} \oplus \mathbb{C}(1 - \delta_{i,0})Y_{i-d_2-\frac{1}{2}} \oplus \mathbb{C}L_i) \oplus \mathbb{C}C.$$

Then, it is easy to see that \mathcal{G}_{d_1, d_2} is a subalgebra of \mathcal{G} .

Letting V be a simple \mathcal{G}_{d_1, d_2} -module, then we have the induced \mathcal{G} -module

$$\text{Ind}(V) = \mathcal{U}(\mathcal{G}) \otimes_{\mathcal{U}(\mathcal{G}_{d_1, d_2})} V.$$

Since simple modules over one of subalgebras of \mathcal{G} containing the central elements M_0 and C are usually considered in the following, we always assume that the actions of M_0 and C are scalars ν_0 and c respectively.

Fix $d_1, d_2 \in \mathbb{N}$ and let V be a simple \mathcal{G}_{d_1, d_2} -module. For $\underline{i}, \underline{j}, \underline{k} \in \mathbb{M}$, we denote

$$M^{\underline{i}} Y^{\underline{j}} L^{\underline{k}} = \dots M_{-d_1-2}^{i_2} M_{-d_1-1}^{i_1} \dots Y_{-d_2-\frac{3}{2}}^{j_2} Y_{-d_2-\frac{1}{2}}^{j_1} \dots L_{-2}^{k_2} L_{-1}^{k_1} \in \mathcal{U}(\mathcal{G}).$$

According to the PBW Theorem, every element of $\text{Ind}(V)$ can be uniquely written in the following form

$$\sum_{\underline{i}, \underline{j}, \underline{k} \in \mathbb{M}} M^{\underline{i}} Y^{\underline{j}} L^{\underline{k}} v_{\underline{i}, \underline{j}, \underline{k}}, \quad (2.1)$$

where all $v_{\underline{i}, \underline{j}, \underline{k}} \in V$ and only finitely many of them are nonzero. For any $v \in \text{Ind}(V)$ as in (2.1), we denote by $\text{supp}(v)$ the set of all $(\underline{i}, \underline{j}, \underline{k}) \in \mathbb{M} \times \mathbb{M} \times \mathbb{M}$ such that $v_{\underline{i}, \underline{j}, \underline{k}} \neq 0$. For a nonzero $v \in \text{Ind}(V)$, we write $\deg(v)$ the maximal (with respect to the principal total order on $\mathbb{M} \times \mathbb{M} \times \mathbb{M}$) element in $\text{supp}(v)$, called the *degree* of v . Note that here and later we make the convention that $\deg(v)$ only for $v \neq 0$.

3 Characterization of simple modules

The purpose of this section is to state two main results of this paper. We first prove that the induced \mathcal{G} -module $\text{Ind}(V)$ is simple under certain conditions which appeared in Theorem 3.1. Then we shall show that under the conditions in Theorem 3.1 any simple \mathcal{G} -module with locally finite actions of elements $M_i, (1 - \delta_{j,0})Y_{j-\frac{1}{2}}, L_k$ for sufficiently large $i, j, k \in \mathbb{N}$ is isomorphic to one of the induced \mathcal{G} -modules $\text{Ind}(V)$.

Now we can summarize the key result in this section as follows.

Theorem 3.1. *Let $d_1, d_2 \in \mathbb{N}$ and V be a simple \mathcal{G}_{d_1, d_2} -module and there exists $t \in \mathbb{N}$ satisfying the following two conditions:*

- (a) *the action of M_t on V is injective;*
- (b) *$M_i V = Y_{j-\frac{1}{2}} V = L_k V = 0$ for all $i > t, j > t + d_2$ and $k > t + d_1$.*

Then we have

- (1) *$\text{Ind}(V)$ is a simple \mathcal{G} -module;*
- (2) *the actions of $M_i, Y_{j-\frac{1}{2}}, L_k$ on $\text{Ind}(V)$ for all $i > t, j > t + d_2$ and $k > t + d_1$ are locally nilpotent.*

Proof. In order to prove part (1) of Theorem 3.1, we first introduce the following claim.

Claim 1. *For any $v \in \text{Ind}(V) \setminus V$, let $\deg(v) = (\underline{i}, \underline{j}, \underline{k})$, $\hat{i} = \max\{s : i_s \neq 0\}$ if $\underline{i} \neq \underline{0}$, $\hat{j} = \min\{s : j_s \neq 0\}$ if $\underline{j} \neq \underline{0}$ and $\hat{k} = \min\{s : k_s \neq 0\}$ if $\underline{k} \neq \underline{0}$. Then we obtain*

- (1) if $\underline{k} \neq \underline{0}$, then $\hat{k} > 0$ and $\deg(M_{\hat{k}+t}v) = (\underline{i}, \underline{j}, \underline{k}'')$;
(2) if $\underline{k} = \underline{0}, \underline{j} \neq \underline{0}$, then $\hat{j} > 0$ and $\deg(Y_{\hat{j}+t+d_2-\frac{1}{2}}v) = (\underline{i}, \underline{j}'', 0)$;
(3) if $\underline{j} = \underline{k} = \underline{0}, \underline{i} \neq \underline{0}$, then $\hat{i} > 0$ and $\deg(L_{\hat{i}+t+d_1}v) = (\underline{i}', 0, 0)$.

To prove this, we assume that v is of the form in (2.1).

(1) It is enough to show that we want to have by comparing the degree. Now we consider those $v_{\underline{x}, \underline{y}, \underline{z}}$ with

$$M_{\hat{k}+t}M^{\underline{x}}Y^{\underline{y}}L^{\underline{z}}v_{\underline{x}, \underline{y}, \underline{z}} \neq 0.$$

Note that $M_{\hat{k}+t}v_{\underline{x}, \underline{y}, \underline{z}} = 0$ for any $(\underline{x}, \underline{y}, \underline{z}) \in \text{supp}(v)$. One can easily check that

$$M_{\hat{k}+t}M^{\underline{x}}Y^{\underline{y}}L^{\underline{z}}v_{\underline{x}, \underline{y}, \underline{z}} = M^{\underline{x}}Y^{\underline{y}}[M_{\hat{k}+t}, L^{\underline{z}}]v_{\underline{x}, \underline{y}, \underline{z}}.$$

By (a), $M_t v_{\underline{x}, \underline{y}, \underline{z}} \neq 0$. If $\underline{z} = \underline{k}$, it is easy to see that

$$\deg(M_{\hat{k}+t}M^{\underline{x}}Y^{\underline{y}}L^{\underline{z}}v_{\underline{x}, \underline{y}, \underline{z}}) = (\underline{x}, \underline{y}, \underline{k}'') \preceq (\underline{i}, \underline{j}, \underline{k}''),$$

where the equality holds if and only if $\underline{y} = \underline{j}, \underline{x} = \underline{i}$.

Now we suppose $(\underline{z}, \mathbf{w}(\underline{z})) \prec (\underline{k}, \mathbf{w}(\underline{k}))$ and denote

$$\deg(M_{\hat{k}+t}M^{\underline{x}}Y^{\underline{y}}L^{\underline{z}}v_{\underline{x}, \underline{y}, \underline{z}}) = (\underline{x}_1, \underline{y}_1, \underline{z}_1) \in \mathbb{M} \times \mathbb{M} \times \mathbb{M}.$$

If $\mathbf{w}(\underline{z}) < \mathbf{w}(\underline{k})$, then we get $\mathbf{w}(\underline{z}_1) \leq \mathbf{w}(\underline{z}) - \hat{k} < \mathbf{w}(\underline{k}) - \hat{k} = \mathbf{w}(\underline{k}'')$, which gives rise to $(\underline{x}_1, \underline{y}_1, \underline{z}_1) \prec (\underline{i}, \underline{j}, \underline{k}'')$.

Then we suppose $\mathbf{w}(\underline{z}) = \mathbf{w}(\underline{k})$ and $\underline{z} \prec \underline{k}$. Let $\hat{z} := \min\{s : z_s \neq 0\} > 0$. If $\hat{z} > \hat{k}$, it is easy to see that $\mathbf{w}(\underline{z}_1) < \mathbf{w}(\underline{z}) - \hat{k} = \mathbf{w}(\underline{k}'')$. If $\hat{z} = \hat{k}$, we can similarly deduce $(\underline{x}_1, \underline{y}_1, \underline{z}_1) = (\underline{x}, \underline{y}, \underline{z}'')$. Since $\underline{z}'' \prec \underline{k}''$, we have $\deg(M_{\hat{k}+t}M^{\underline{x}}Y^{\underline{y}}L^{\underline{z}}v_{\underline{x}, \underline{y}, \underline{z}}) = (\underline{x}_1, \underline{y}_1, \underline{z}_1) \prec (\underline{i}, \underline{j}, \underline{k}'')$ in both cases.

Combining all the arguments above we conclude that $\deg(M_{\hat{k}+t}v) = (\underline{i}, \underline{j}, \underline{k}'')$, as desired.

(2) Now we use the similar method that appeared in above. We consider $v_{\underline{x}, \underline{y}, \underline{0}}$ with

$$Y_{\hat{j}+t+d_2-\frac{1}{2}}M^{\underline{x}}Y^{\underline{y}}v_{\underline{x}, \underline{y}, \underline{0}} \neq 0.$$

Since $Y_{\hat{j}+t+d_2-\frac{1}{2}}v_{\underline{x}, \underline{y}, \underline{0}} = 0$ for any $(\underline{x}, \underline{y}, \underline{0}) \in \text{supp}(v)$, then we have

$$Y_{\hat{j}+t+d_2-\frac{1}{2}}M^{\underline{x}}Y^{\underline{y}}v_{\underline{x}, \underline{y}, \underline{0}} = M^{\underline{x}}[Y_{\hat{j}+t+d_2-\frac{1}{2}}, Y^{\underline{y}}]v_{\underline{x}, \underline{y}, \underline{0}}.$$

Note that $M_t v_{\underline{x}, \underline{y}, \underline{0}} \neq 0$. If $\underline{y} = \underline{j}$, it is easy to get that

$$\deg(Y_{\hat{j}+t+d_2-\frac{1}{2}}M^{\underline{x}}Y^{\underline{y}}v_{\underline{x}, \underline{y}, \underline{0}}) = (\underline{x}, \underline{y}'', \underline{0}) \preceq (\underline{i}, \underline{j}'', \underline{0}),$$

where the equality holds if and only if $\underline{x} = \underline{i}$.

Now suppose $(\underline{y}, \mathbf{w}(\underline{y})) \prec (\underline{j}, \mathbf{w}(\underline{j}))$, then we write

$$\deg(Y_{\hat{j}+t+d_2-\frac{1}{2}} M^{\underline{x}} Y^{\underline{y}} v_{\underline{x}, \underline{y}, \underline{0}}) = (\underline{x}_1, \underline{y}_1, \underline{0}) \in \mathbb{M} \times \mathbb{M} \times \mathbb{M}.$$

If $\mathbf{w}(\underline{y}) < \mathbf{w}(\underline{j})$, then we get $\mathbf{w}(\underline{y}_1) \leq \mathbf{w}(\underline{y}) - \hat{j} < \mathbf{w}(\underline{j}) - \hat{j} = \mathbf{w}(\underline{j}'')$, which shows that $(\underline{x}_1, \underline{y}_1, \underline{0}) \prec (\underline{i}, \underline{j}'', \underline{0})$.

Then we suppose $\mathbf{w}(\underline{y}) = \mathbf{w}(\underline{j})$ and $\underline{y} \prec \underline{j}$. Let $\hat{y} := \min\{s : y_s \neq 0\} > 0$. If $\hat{y} > \hat{j}$, we obtain $\mathbf{w}(\underline{y}_1) < \mathbf{w}(\underline{y}) - \hat{j} = \mathbf{w}(\underline{j}'')$. If $\hat{y} = \hat{j}$, we can similarly check that $(\underline{x}_1, \underline{y}_1, \underline{0}) = (\underline{x}, \underline{y}'', \underline{0})$. By $\underline{y}'' \prec \underline{j}''$, we have $\deg(Y_{\hat{j}+t+d_2-\frac{1}{2}} M^{\underline{x}} Y^{\underline{y}} v_{\underline{x}, \underline{y}, \underline{0}}) = (\underline{x}_1, \underline{y}_1, \underline{0}) \prec (\underline{i}, \underline{j}'', \underline{0})$ in both cases.

Consequently, we conclude that $\deg(Y_{\hat{j}+t+d_2-\frac{1}{2}} v) = (\underline{i}, \underline{j}'', \underline{0})$.

(3) Noticing that $L_{\hat{i}+t+d_1} V = 0$ and $[L_{\hat{i}+t+d_1}, M_{-\hat{i}-d_1}] V \neq 0$, then we have

$$L_{\hat{i}+t+d_1} M^{\underline{x}} v_{\underline{x}, \underline{0}, \underline{0}} = a_{\hat{i}} M^{\underline{x}'} [L_{\hat{i}+t+d_1}, M_{-\hat{i}-d_1}] v_{\underline{x}, \underline{0}, \underline{0}} \quad \text{for } a_{\hat{i}} \in \mathbb{C} \setminus \{0\}, \underline{x}' \in \mathbb{M}, (\underline{x}, \underline{0}, \underline{0}) \in \text{supp}(v),$$

which easily yields our result. This proves Claim 1.

Using Claim 1 repeatedly, from any nonzero element $v \in \text{Ind}(V)$ we can reach a nonzero element in $\mathcal{U}(\mathcal{G})v \cap V \neq 0$, which indicates the simplicity of $\text{Ind}(V)$. Part (2) of Theorem 3.1 can be easily checked by a direct calculation. \square

Remark 3.2. In Theorem 3.1, when $t = 0$, the condition (a) is equivalent to that $\nu_0 \neq 0$. In addition, from the above proof, we see that Claim 1 also holds without the assumption of the simplicity of V as a \mathcal{G}_{d_1, d_2} -module.

Moreover, we have the following corollary.

Corollary 3.3. *Letting d_1, d_2 and V as in Theorem 3.1 except that V may not be simple over \mathcal{G}_{d_1, d_2} , then we have*

$$V = \{v \in \text{Ind}(V) \mid M_i v = Y_{j-\frac{1}{2}} v = L_i v = 0, \forall i > t, j > t + d_2, k > t + d_1\}.$$

Denote by $\mathcal{G}^{(x, y, z)}$ the subalgebra generated by $M_i, (1 - \delta_{j,0})Y_{j-\frac{1}{2}}, L_k$ with $i \geq x, j \geq y$ and $k \geq z$. Now we are ready to state the second main result of this section.

Theorem 3.4. *Let $\nu_0 \neq 0$ and S be a simple \mathcal{G} -module. Then the following conditions are equivalent:*

- (1) *There exists $t \in \mathbb{Z}$ such that the actions of $M_i, (1 - \delta_{i,0})Y_{i-\frac{1}{2}}, L_i, i \geq t$ on S are locally finite.*
- (2) *There exists $t \in \mathbb{Z}$ such that the actions of $M_i, (1 - \delta_{i,0})Y_{i-\frac{1}{2}}, L_i, i \geq t$ on S are locally nilpotent.*

- (3) There exist $x, y, z \in \mathbb{Z}$ such that S is a locally finite $\mathcal{G}^{(x,y,z)}$ -module.
- (4) There exist $x, y, z \in \mathbb{Z}$ such that S is a locally nilpotent $\mathcal{G}^{(x,y,z)}$ -module.
- (5) There exist $d_1, d_2 \in \mathbb{N}$ and a simple \mathcal{G}_{d_1, d_2} -module V satisfying the conditions in Theorem 3.1 such that $S \cong \text{Ind}(V)$.

Proof. First we prove (1) \Rightarrow (5). Suppose that S is a simple \mathcal{G} -module and there exists $t \in \mathbb{N}$ such that the actions of $M_i, (1 - \delta_{i,0})Y_{i-\frac{1}{2}}$, and L_i for all $i \geq t$ are locally finite. Then we can choose a nonzero $v \in S$ such that $L_t v = \lambda v$ for some $\lambda \in \mathbb{C}$.

Take any $j \in \mathbb{Z}$ with $j > t$ and we denote

$$\begin{aligned} N_M &= \sum_{m \in \mathbb{N}} \mathbb{C} L_t^m M_j v = \mathcal{U}(\mathbb{C} L_t) M_j v, \\ N_Y &= \sum_{m \in \mathbb{N}} \mathbb{C} L_t^m Y_{j-\frac{1}{2}} v = \mathcal{U}(\mathbb{C} L_t) Y_{j-\frac{1}{2}} v, \\ N_L &= \sum_{m \in \mathbb{N}} \mathbb{C} L_t^m L_j v = \mathcal{U}(\mathbb{C} L_t) L_j v, \end{aligned}$$

which are all finite-dimensional. By the definition of (1.1), it is easy to get

$$\begin{aligned} (j + mt) M_{j+(m+1)t} v &= [L_t, M_{j+mt}] v \\ &= L_t M_{j+mt} v - M_{j+mt} L_t v = (L_t - \lambda) M_{j+mt} v, \\ (j + mt - \frac{t+1}{2}) Y_{j+(m+1)t-\frac{1}{2}} v &= [L_t, Y_{j+mt-\frac{1}{2}}] v \\ &= L_t Y_{j+mt-\frac{1}{2}} v - Y_{j+mt-\frac{1}{2}} L_t v = (L_t - \lambda) Y_{j+mt-\frac{1}{2}} v, \\ (j + (m-1)t) L_{j+(m+1)t} v &= [L_t, L_{j+mt}] v \\ &= L_t L_{j+mt} v - L_{j+mt} L_t v = (L_t - \lambda) L_{j+mt} v, \quad \forall m \in \mathbb{N}, \end{aligned}$$

which imply that

$$\begin{aligned} M_{j+mt} v \in N_M &\Rightarrow M_{j+(m+1)t} v \in N_M, \quad Y_{j+mt-\frac{1}{2}} v \in N_Y \Rightarrow Y_{j+(m+1)t-\frac{1}{2}} v \in N_Y, \\ L_{j+mt} v \in N_L &\Rightarrow L_{j+(m+1)t} v \in N_L \quad \text{for all } m \in \mathbb{N} \text{ and } j > t. \end{aligned}$$

Therefore, by induction on m , we obtain

$$M_{j+mt} v \in N_M, \quad Y_{j+mt-\frac{1}{2}} v \in N_Y, \quad L_{j+mt} v \in N_L, \quad \forall m \in \mathbb{N}.$$

Then, it follows from the facts that $\sum_{m \in \mathbb{N}} \mathbb{C} M_{j+mt} v$, $\sum_{m \in \mathbb{N}} \mathbb{C} Y_{j+mt-\frac{1}{2}} v$ and $\sum_{m \in \mathbb{N}} \mathbb{C} L_{j+mt} v$

are finite-dimensional for $j > t$. Hence,

$$\begin{aligned}\sum_{i \in \mathbb{N}} \mathbb{C}M_{t+i}v &= \mathbb{C}M_tv + \sum_{j=t+1}^{2t} \left(\sum_{m \in \mathbb{N}} \mathbb{C}M_{j+mt}v \right), \\ \sum_{i \in \mathbb{N}} \mathbb{C}Y_{t+i-\frac{1}{2}}v &= \mathbb{C}Y_{t-\frac{1}{2}}v + \sum_{j=t+1}^{2t} \left(\sum_{m \in \mathbb{N}} \mathbb{C}Y_{j+mt-\frac{1}{2}}v \right), \\ \sum_{i \in \mathbb{N}} \mathbb{C}L_{t+i}v &= \mathbb{C}L_tv + \sum_{j=t+1}^{2t} \left(\sum_{m \in \mathbb{N}} \mathbb{C}L_{j+mt}v \right)\end{aligned}$$

are all finite-dimensional. In fact, we can take $l \in \mathbb{Z}_+$ such that

$$\sum_{i \in \mathbb{N}} \mathbb{C}M_{t+i}v = \sum_{i=0}^l \mathbb{C}M_{t+i}v, \quad \sum_{i \in \mathbb{N}} \mathbb{C}Y_{t+i-\frac{1}{2}}v = \sum_{i=0}^l \mathbb{C}Y_{t+i-\frac{1}{2}}v, \quad \sum_{i \in \mathbb{N}} \mathbb{C}L_{t+i}v = \sum_{i=0}^l \mathbb{C}L_{t+i}v. \quad (3.1)$$

Now we write $V' = \sum_{x_0, \dots, x_l, y_0, \dots, y_l, z_0, \dots, z_l \in \mathbb{N}} \mathbb{C}M_t^{x_0} \dots M_{t+l}^{x_l} Y_{t-\frac{1}{2}}^{y_0} \dots Y_{t+l-\frac{1}{2}}^{y_l} L_t^{z_0} \dots L_{t+l}^{z_l} v$, which is finite-dimensional by (1).

Claim 1. V' is a (finite-dimensional) $\mathcal{G}^{(t,t,t)}$ -module.

To prove the claim, using the PBW Theorem, any $M_{t+s}v', Y_{t+s-\frac{1}{2}}v', L_{t+s}v'$ with $s \in \mathbb{N}$ and $v' \in V'$ can be written respectively as a sum of vectors of the form:

$$\begin{aligned}M_{t+s}M_t^{x_0} \dots M_{t+l}^{x_l} Y_{t-\frac{1}{2}}^{y_0} \dots Y_{t+l-\frac{1}{2}}^{y_l} L_t^{z_0} \dots L_{t+l}^{z_l} v, \\ Y_{t+s-\frac{1}{2}}M_t^{x_0} \dots M_{t+l}^{x_l} Y_{t-\frac{1}{2}}^{y_0} \dots Y_{t+l-\frac{1}{2}}^{y_l} L_t^{z_0} \dots L_{t+l}^{z_l} v, \\ L_{t+s}M_t^{x_0} \dots M_{t+l}^{x_l} Y_{t-\frac{1}{2}}^{y_0} \dots Y_{t+l-\frac{1}{2}}^{y_l} L_t^{z_0} \dots L_{t+l}^{z_l} v.\end{aligned} \quad (3.2)$$

Now we prove that all elements above lie in V' . By (3.1), we only need to show that the elements in (3.2) with $0 \leq s \leq l$ lie in V' . This is clear for the first element in (3.2). For the second element in (3.2), we have

$$\begin{aligned}Y_{t+s-\frac{1}{2}}M_t^{x_0} \dots M_{t+l}^{x_l} Y_{t-\frac{1}{2}}^{y_0} \dots Y_{t+l-\frac{1}{2}}^{y_l} L_t^{z_0} \dots L_{t+l}^{z_l} v \\ = M_t^{x_0} \dots M_{t+l}^{x_l} Y_{t-\frac{1}{2}}^{y_0} \dots Y_{t+s-\frac{1}{2}}^{y_s+1} \dots Y_{t+l-\frac{1}{2}}^{y_l} L_t^{z_0} \dots L_{t+l}^{z_l} v \\ + [Y_{t+s-\frac{1}{2}}, M_t^{x_0} \dots M_{t+l}^{x_l} Y_{t-\frac{1}{2}}^{y_0} \dots Y_{t+s-\frac{1}{2}}^{y_s}] Y_{t+s+\frac{1}{2}}^{y_s+1} \dots Y_{t+l-\frac{1}{2}}^{y_l} L_t^{z_0} \dots L_{t+l}^{z_l} v.\end{aligned}$$

Then, it is clear that the second element in (3.2) lies in V' . For the third element in (3.2),

one can easily check that

$$\begin{aligned}
& L_{t+s} M_t^{x_0} \cdots M_{t+l}^{x_l} Y_{t-\frac{1}{2}}^{y_0} \cdots Y_{t+l-\frac{1}{2}}^{y_l} L_t^{z_0} \cdots L_{t+l}^{z_l} v \\
&= M_t^{x_0} \cdots M_{t+l}^{x_l} Y_{t-\frac{1}{2}}^{y_0} \cdots Y_{t+l-\frac{1}{2}}^{y_l} L_t^{z_0} \cdots L_{t+s}^{z_s+1} \cdots L_{t+l}^{z_l} v \\
&\quad + [L_{t+s}, M_t^{x_0} \cdots M_{t+l}^{x_l} Y_{t-\frac{1}{2}}^{y_0} \cdots Y_{t+l-\frac{1}{2}}^{y_l} L_t^{z_0} \cdots L_{t+s}^{z_s}] L_{t+s+1}^{z_s+1} \cdots L_{t+l}^{z_l} v.
\end{aligned}$$

Using induction, we can show that all terms in above equation lie in V' . Hence we can get the third element in (3.2) also lies in V' . Then Claim 1 is obtained.

It follows from Claim 1 that we can choose a minimal $n \in \mathbb{N}$ such that

$$(L_m + a_1 L_{m+1} + \cdots + a_n L_{m+n}) V' = 0 \quad (3.3)$$

for some $m \geq t$ and $a_i \in \mathbb{C}$. Applying L_m to (3.3), one has

$$(a_1 [L_m, L_{m+1}] + \cdots + a_n [L_m, L_{m+n}]) V' = 0,$$

which implies $n = 0$, that is, $L_m V' = 0$. Then we have

$$0 = L_i L_m V' = [L_i, L_m] V' + L_m L_i V' = (m - i) L_{m+i} V', \quad \forall i \geq t,$$

namely, $L_{m+i} V' = 0$ for all $i > m$. Similarly, we have $M_{m+i} V' = 0$ for all $i > t$ and $Y_{m+i-\frac{1}{2}} V' = 0$ for all $1 \neq i > m$, respectively. For any $\tilde{i}, \tilde{j}, \tilde{k} \in \mathbb{Z}$, we consider the vector space

$$N_{\tilde{i}, \tilde{j}, \tilde{k}} = \{v \in S \mid M_i v = (1 - \delta_{j,0}) Y_{j-\frac{1}{2}} v = L_k v = 0 \text{ for all } i > \tilde{i}, j > \tilde{j}, k > \tilde{k}\}.$$

Clearly, $N_{\tilde{i}, \tilde{j}, \tilde{k}} \neq 0$ for sufficiently large $\tilde{i}, \tilde{j}, \tilde{k} \in \mathbb{Z}$. On the other hand, $N_{\tilde{i}, \tilde{j}, \tilde{k}} = 0$ for all $\tilde{i} < 0$ since we have $M_0 v = \nu_0 v \neq 0$ for any nonzero $v \in S$. Thus we can find a smallest nonnegative integer, saying r_1 , and choose some $r_2, r_3 \geq r_1$ with $r_3 - r_1 \geq 2(r_2 - r_1) - 1$ such that $N_{r_1, r_2, r_3} \neq 0$. Denote $d_1 = r_3 - r_1, d_2 = r_2 - r_1$ and $V = N_{r_1, r_2, r_3}$. Using $k > r_3, j > r_2$ and $l \geq 1$, it follows from $k + l - d_2 - \frac{1}{2} > r_3 + \frac{1}{2} - d_2 \geq r_2 - \frac{1}{2}$ and $j + l - d_2 - 1 > r_2 - d_2 = r_1$ that we can easily check that

$$L_k (Y_{l-d_2-\frac{1}{2}} v) = (l - d_2 - \frac{k+1}{2}) Y_{k+l-d_2-\frac{1}{2}} v = 0$$

and

$$Y_{j-\frac{1}{2}} (Y_{l-d_2-\frac{1}{2}} v) = (l - d_2 - j) M_{j+l-d_2-1} v = 0,$$

respectively. Clearly, $Y_{l-d_2-\frac{1}{2}} v \in V$ for all $l \geq 1$. Similarly, we can also obtain $M_{e-d_1} v \in V$ and $L_e v \in V$ for all $e \in \mathbb{N}$. Therefore, V is a \mathcal{G}_{d_1, d_2} -module.

By the definition of V , we can obtain that the action of M_t on V is injective. Since S is simple and generated by V , then there exists a canonical surjective map

$$\pi : \text{Ind}(V) \rightarrow S, \quad \pi(1 \otimes v) = v, \quad \forall v \in V.$$

Next we only need to show that π is also injective, that is to say, π as the canonical map is bijective. Let $K = \ker(\pi)$. Obviously, $K \cap V = 0$. If $K \neq 0$, we can choose a nonzero vector $v \in K \setminus V$ such that $\deg(v) = (\underline{i}, \underline{j}, \underline{k})$ is minimal possible. Note that K is a \mathcal{G} -submodule of $\text{Ind}(V)$. By Claim 1 in Theorem 3.1 and Remark 3.2 we can create a new vector $u \in K$ with $\deg(u) \prec (\underline{i}, \underline{j}, \underline{k})$, which is a contradiction. This forces $K = 0$, that is, $S \cong \text{Ind}(V)$. According to the property of induced modules, we see that V is simple as a \mathcal{G}_{d_1, d_2} -module.

Moreover, (5) \Rightarrow (3) \Rightarrow (1), (5) \Rightarrow (4) \Rightarrow (2) and (2) \Rightarrow (1) are clear. This completes the proof of the theorem. \square

Remark 3.5. From the above proof, we know that any simple module satisfying conditions in Theorem 3.4 is determined by some simple module V over a certain subalgebra \mathcal{G}_{d_1, d_2} . The conditions of Theorem 3.1 imply that V can be viewed as a simple module over some finite-dimensional solvable quotient algebra of \mathcal{G}_{d_1, d_2} . This reduces the study of such modules over \mathcal{G} to the study of simple modules over the corresponding finite-dimensional algebras.

4 Some examples

In this section, some examples of simple \mathcal{G}_{d_1, d_2} -modules are given. Then, by Theorem 3.1, we can construct many new simple \mathcal{G} -modules.

First we describe highest weight modules and Whittaker modules as follows.

Example 4.1. Let $\mathfrak{h} = \text{span}_{\mathbb{C}}\{L_0, M_0, C\}$ be the Cartan subalgebra of \mathcal{G} . For $\xi = (\xi(L_0), \nu_0 \neq 0, c) \in \mathfrak{h}^*$, we have the Verma module $\mathcal{M}(\xi) = \mathcal{U}(\mathcal{G}) \otimes_{\mathcal{U}(\mathfrak{h} + \mathcal{G}_{0,0})} \mathbb{C}_{\xi}$, where $L_i \mathbb{C}_{\xi} = M_i \mathbb{C}_{\xi} = Y_{i-\frac{1}{2}} \mathbb{C}_{\xi}$ for $i > 0$, while $L_0 \mathbb{C}_{\xi} = \xi(L_0) \mathbb{C}_{\xi}$, $M_0 \mathbb{C}_{\xi} = \nu_0 \mathbb{C}_{\xi}$ and $C \mathbb{C}_{\xi} = c \mathbb{C}_{\xi}$, respectively. The module $\mathcal{M}(\xi)$ has the unique simple quotient $\mathcal{L}(\xi)$, the unique (up to isomorphism) simple highest weight module with highest weight ξ . These modules correspond to the case $t = 0$ in Theorem 3.4.

Example 4.2. Consider a nonzero $\xi := (\lambda_1, \lambda_2, \mu_1, \mu_2, \nu_0 \neq 0, \nu_1 \neq 0, c) \in \mathbb{C}^7$. Denote by \mathcal{V}_{ξ} the $\mathcal{G}_{1,1}$ -module $\mathcal{U}(\mathcal{G}_{1,1})/I$, where I is the left ideal generated by $L_1 - \lambda_1, L_2 - \lambda_2, L_3, \dots, Y_{\frac{1}{2}} - \mu_1, Y_{\frac{3}{2}} - \mu_2, Y_{\frac{5}{2}}, \dots, M_1 - \nu_1, M_2, \dots, M_0 - \nu_0, C - c$. It is easy to see that \mathcal{V}_{ξ} is simple. The module \mathcal{V}_{ξ} obviously satisfies the conditions of Theorem 3.1 (with $t = d_1 = d_2 = 1$). Hence, it follows from Theorem 3.1 that we obtain the corresponding simple induced \mathcal{G} -module $\text{Ind}(\mathcal{V}_{\xi})$. When $c = 0$, these are exactly the Whittaker modules over \mathcal{G} constructed in [24].

Now we consider some $t \in \mathbb{Z}_+$, $d_1, d_2 \in \mathbb{N}$, and choose subsets $S_\lambda \subseteq \{1, \dots, t + d_1\}$, $S_\mu \subseteq \{-d_2 + 1, \dots, t + d_2\}$ and disjoint subsets $S_{\nu,0}, S_{\nu,1} \subseteq \{-d_1, \dots, t\}$ with $0, t \in S_{\nu,1}$. Set $S_\nu = S_{\nu,0} \cup S_{\nu,1}$. Let $\bar{S}_\lambda = \{0, 1, \dots, t + d_1\} \setminus S_\lambda$, $\bar{S}_\mu = \{-d_2, \dots, t + d_2\} \setminus S_\mu$ and $\bar{S}_\nu = \{-d_1, \dots, t\} \setminus S_\nu$. Set $c \in \mathbb{C}$, $\lambda_i \in \mathbb{C}, i \in S_\lambda, \mu_i \in \mathbb{C}, i \in S_\mu$ and $\nu_i \in \mathbb{C}, i \in S_\nu$. Moreover, $i \in S_\nu$ with $\nu_i \neq 0$ if and only if $i \in S_{\nu,1}$. In addition, the following conditions are satisfied:

- (I) for all $i, j \in S_\lambda, i \neq j$, we either have $i + j > t + d_1$ or $i + j \in S_\lambda$ and $\lambda_{i+j} = 0$;
- (II) for all $i \in S_\lambda, j \in S_\mu, \frac{i}{2} \neq j - \frac{1}{2}$, we either have $i + j > t + d_2$ or $i + j \in S_\mu$ and $\mu_{i+j} = 0$;
- (III) for all $i, j \in S_\mu, i \neq j$, we either have $i + j - 1 > t$ or $i + j - 1 \in S_{\nu,0}$ and $\nu_{i+j-1} = 0$;
- (IV) for all $i \in S_\lambda, j \in S_\nu \setminus \{0\}$, we either have $i + j > t$ or $i + j \in S_{\nu,0}$ and $\nu_{i+j} = 0$;
- (V) for any $j \in \bar{S}_\lambda$, there exists a nonzero $i \in S_\nu$ such that $i + j \in \bar{S}_\nu \cup S_{\nu,1}$ and $i + j' \in S_{\nu,0}$ for all $j' \in \bar{S}_\lambda$ with $j < j' < t - i$ (if $t - i \in \bar{S}_\lambda$ we replace $t - i$ by j and these j such that $i + j \in S_{\nu,1}$);
- (VI) for any $j \in \bar{S}_\mu$, there exists $i \in S_\mu$ such that $i + j - 1 \in \bar{S}_\nu \cup S_{\nu,1}$ and $i + j' - 1 \in S_{\nu,0}$ for all $j' \in \bar{S}_\mu$ with $j < j' < t - i + 1$ (if $t - i + 1 \in \bar{S}_\mu$ we replace $t - i + 1$ by j and these j such that $i + j - 1 \in S_{\nu,1}$);
- (VII) for any $j \in \bar{S}_\nu$, there exists $i \in S_\lambda$ such that $i + j \in \bar{S}_\nu \cup S_{\nu,1}$ and $i + j' \in S_{\nu,0}$ for all $j' \in \bar{S}_\nu$ with $j < j' < t - i$ (if $t - i \in \bar{S}_\nu$ we replace $t - i$ by j and these j such that $i + j \in S_{\nu,1}$).

For any $\underline{i} = (i_1, \dots, i_t), \underline{j} = (j_1, \dots, j_t), \underline{k} = (k_1, \dots, k_t) \in \mathbb{N}^t, t \in \mathbb{Z}_+$, we can define the lexicographical order (which is *not* the one defined in Section 2) on \mathbb{N}^t as follows:

$$\underline{i} > \underline{j} \Leftrightarrow \text{there exists } r \text{ such that } (i_s = j_s, \forall 1 \leq s < r) \text{ and } i_r > j_r.$$

Denote $\epsilon_r \in \mathbb{N}^t$ with 1 in the r 'th position and 0 elsewhere.

For any set $X = \{x_1, \dots, x_n\}$, we denote by $|X|$ the number of elements in X . Now we set $m = |\bar{S}_\lambda|, n = |\bar{S}_\mu|, l = |\bar{S}_\nu|$. Let $\bar{S}_\lambda = \{p_1 = 0, p_2, \dots, p_m\}$ with $p_1 < \dots < p_m$, $\bar{S}_\mu = \{q_1 = -d_2, q_2, \dots, q_n\}$ with $q_1 < \dots < q_n$ and $\bar{S}_\nu = \{r_1, r_2, \dots, r_l\}$ with $r_1 < \dots < r_l$. Denote by Q the \mathcal{G}_{d_1, d_2} -module $\mathcal{U}(\mathcal{G}_{d_1, d_2})/I$, where I is the left ideal generated by $L_i - \lambda_i, Y_{j-\frac{1}{2}} - \mu_j, M_k - \nu_k, C - c$ with $i \in \mathbb{N} \setminus \bar{S}_\lambda, j \in (\mathbb{N} - d_2) \setminus \bar{S}_\mu, k \in (\mathbb{N} - d_1) \setminus \bar{S}_\nu$, where we make the convention that $\lambda_i = 0$ for $i \notin S_\lambda, \mu_j = 0$ for $j \notin S_\mu$ and $\nu_k = 0$ for $k \notin S_{\nu,1}$. By the PBW Theorem, a basis of Q is given by the images of

$$L^{\underline{i}} Y^{\underline{j}} M^{\underline{k}} = L_{p_1}^{i_1} \dots L_{p_m}^{i_m} Y_{q_1 - \frac{1}{2}}^{j_1} \dots Y_{q_n - \frac{1}{2}}^{j_n} M_{r_1}^{k_1} \dots M_{r_l}^{k_l},$$

where $\underline{i} := (i_1, \dots, i_m) \in \mathbb{N}^m, \underline{j} := (j_1, \dots, j_n) \in \mathbb{N}^n, \underline{k} := (k_1, \dots, k_l) \in \mathbb{N}^l$. Then a typical element $v \in Q$ can be written as

$$v = \sum_{\underline{i} \in \mathbb{N}^m, \underline{j} \in \mathbb{N}^n, \underline{k} \in \mathbb{N}^l} a_{\underline{i}, \underline{j}, \underline{k}} L^{\underline{i}} Y^{\underline{j}} M^{\underline{k}}, \quad (4.1)$$

with only finitely many $a_{\underline{i}, \underline{j}, \underline{k}}$ nonzero. Set $\text{supp}(v) = \{(\underline{i}, \underline{j}, \underline{k}) \mid a_{\underline{i}, \underline{j}, \underline{k}} \neq 0\}$ and denote by $\deg(v)$ the maximal element in $\text{supp}(v)$ under the following total order

$$(\underline{i}, \underline{j}, \underline{k}) \succ (\underline{l}, \underline{m}, \underline{n}) \Leftrightarrow \underline{i} > \underline{l} \quad \text{or} \quad \underline{i} = \underline{l} \text{ and } \underline{j} > \underline{m} \quad \text{or} \quad \underline{i} = \underline{l}, \underline{j} = \underline{m} \text{ and } \underline{k} > \underline{n}$$

for any $(\underline{i}, \underline{j}, \underline{k}), (\underline{l}, \underline{m}, \underline{n}) \in \mathbb{N}^m \times \mathbb{N}^n \times \mathbb{N}^l$. We also make the convention that the zero element does not have a degree. Now we can prove the following lemma.

Lemma 4.3. *The \mathcal{G}_{d_1, d_2} -module Q is simple.*

Proof. Since \bar{S}_λ is not empty, $Q \neq 0$. By conditions (I)-(IV), Q is a module of \mathcal{G}_{d_1, d_2} . Let $v \in Q \setminus \{0\}$ and we can write v as in (4.1) that is a nonzero linear combination of basis elements. Set $\deg(v) = (\underline{i}, \underline{j}, \underline{k})$.

Case 1. $\underline{i} \neq \underline{0}$. Setting $x := \min\{s > 0 \mid i_s \neq 0\}$, then we have $p_x \in \bar{S}_\lambda$. According to condition (V), there exists nonzero $y \in S_\nu$ such that $p_x + y \in \bar{S}_\nu \cup S_{\nu,1}$ and $y + j' \in S_{\nu,0}$ for all $j' \in \bar{S}_\lambda$ with $p_x < j' < t - y$ (if $t - y \in \bar{S}_\lambda$ we replace $t - y$ by p_x and these p_x such that $p_x + y \in S_{\nu,1}$). Applying $M_y - \nu_y$ to v , which gives

$$\deg((M_y - \nu_y)v) = \begin{cases} (\underline{i} - \epsilon_x, \underline{j}, \underline{k} + \epsilon_s), & \text{if } p_x + y = r_s \in \bar{S}_\nu, \\ (\underline{i} - \epsilon_x, \underline{j}, \underline{k}), & \text{if } p_x + y \in S_{\nu,1}. \end{cases}$$

Case 2. $\underline{i} = \underline{0}$. Setting $x := \min\{s > 0 \mid j_s \neq 0\}$, then we have $q_x \in \bar{S}_\mu$. According to condition (VI), there exists $y \in S_\mu$ such that $q_x + y - 1 \in \bar{S}_\nu \cup S_{\nu,1}$ and $y + j' - 1 \in S_{\nu,0}$ for all $j' \in \bar{S}_\mu$ with $q_x < j' < t - y + 1$ (if $t - y + 1 \in \bar{S}_\mu$ we replace $t - y + 1$ by q_x and these q_x such that $q_x + y - 1 \in S_{\nu,1}$). Applying $Y_{y-\frac{1}{2}} - \mu_y$ to v , which gives

$$\deg((Y_{y-\frac{1}{2}} - \mu_y)v) = \begin{cases} (\underline{0}, \underline{j} - \epsilon_x, \underline{k} + \epsilon_s), & \text{if } q_x + y = r_s \in \bar{S}_\nu, \\ (\underline{0}, \underline{j} - \epsilon_x, \underline{k}), & \text{if } q_x + y \in S_{\nu,1}. \end{cases}$$

Case 3. $\underline{i} = \underline{j} = \underline{0}$. Setting $x := \min\{s > 0 \mid k_s \neq 0\}$, then we have $r_x \in \bar{S}_\nu$. According to condition (VII), there exists $y \in S_\lambda$ such that $r_x + y \in \bar{S}_\nu \cup S_{\nu,1}$ and $y + j'$ for all $j' \in \bar{S}_\nu$ with $r_x < j' < t - y$ (if $t - y \in \bar{S}_\nu$ we replace $t - y$ by r_x and these r_x such that $r_x + y \in S_{\nu,1}$). Applying $L_y - \lambda_y$ to v , which gives

$$\deg((L_y - \lambda_y)v) = \begin{cases} (\underline{0}, \underline{0}, \underline{k} - \epsilon_x + \epsilon_s), & \text{if } r_x + y = r_s \in \bar{S}_\nu, \\ (\underline{0}, \underline{0}, \underline{k} - \epsilon_x), & \text{if } r_x + y \in S_{\nu,1}. \end{cases}$$

Repeating this process inductively (with respect to the degree of v), we can reach a nonzero element with degree $\underline{0}$ and this element can generate the whole module Q . Therefore Q is a simple \mathcal{G}_{d_1, d_2} -module. \square

Remark 4.4. Note that the module Q obviously satisfies the conditions of Theorem 3.1. Then one can obtain the corresponding simple induced \mathcal{G} -module $\text{Ind}(Q)$. It follows from Theorem 3.1 that the actions of all the elements $M_i, Y_{j-\frac{1}{2}}, L_k$ for $i > t, j > t + d_2, k > t + d_1$ are locally nilpotent and the actions of all the elements $M_i, Y_{j-\frac{1}{2}}, L_k$ for $i < -d_1, j < -d_2, k < 0$ are injective and free on $\text{Ind}(Q)$ (hence non-locally finite). Moreover, by using Lemma 4.3, the actions of all the elements $M_i, Y_{j-\frac{1}{2}}, L_k$ for $i \in S_{\nu,0}, j \in S_\mu, k \in S_\lambda$, where $\nu_i = \mu_j = \lambda_k = 0$ are locally nilpotent, and the actions of all the elements $M_i, Y_{j-\frac{1}{2}}, L_k$ for $i \in S_{\nu,1}, j \in S_\mu, k \in S_\lambda$, where $\nu_i, \mu_j, \lambda_k \neq 0$ are locally finite. By these facts, we construct a lot of new simple modules, which are not isomorphic to the simple weight modules and simple Whittaker modules (as far as we know simple \mathcal{G} -modules).

When $d_1, d_2 > 0$ or $t > 1$, the simple modules $\text{Ind}(Q)$ are new simple \mathcal{G} -modules as we stated in Remark 4.4. Finally, we give an example about new simple modules.

Example 4.5. For $d_1 = d_2 = 0$ and $t = 2$, we can define

$$S_\lambda = \{2\}, \bar{S}_\lambda = \{0, 1\}, S_\mu = \{1\}, \bar{S}_\mu = \{0, 2\}, S_{\nu,1} = \{0, 2\}, S_{\nu,0} = \{1\}, \bar{S}_\nu = \emptyset$$

and $\lambda_2, \mu_1, \nu_0, \nu_1, \nu_2, c \in \mathbb{C}$ such that $\nu_1 = 0, \nu_0, \nu_2 \neq 0$. Then, we obtain a $\mathcal{G}_{0,0}$ -module Q . It is easy to see that all conditions (I)-(VII) are satisfied. Then $\mathcal{G}_{0,0}$ -module Q is a simple module. Therefore, the induced module $\text{Ind}(Q)$ is simple, which is a new simple module of \mathcal{G} .

5 The W -algebra $W(2, 2)$

In this section, we shall construct a large class of simple modules over the W -algebra $W(2, 2)$. As a result, we not only recover many known simple modules including highest weight modules and Whittaker modules that presented in [21] and [22], but also construct a lot of new simple modules for the W -algebra $W(2, 2)$.

The W -algebra $W(2, 2)$ \mathcal{W} is defined to be a Lie algebra with a \mathbb{C} -basis $\{L_m, W_m, C_W \mid m \in \mathbb{Z}\}$ and the following nonvanishing Lie brackets:

$$\begin{aligned} [L_m, L_n] &= (n - m)L_{m+n} + \delta_{m+n,0} \frac{1}{12}(m^3 - m)C_W, \\ [L_m, W_n] &= (n - m)W_{m+n} + \delta_{m+n,0} \frac{1}{12}(m^3 - m)C_W. \end{aligned}$$

It was introduced by Zhang and Dong in [22] for the study of the classification of moonshine type vertex operator algebras generated by two weight 2 vectors.

Using lexicographical total order and reverse lexicographical total order defined in Section 2, we induce a *principal total order* (which is *not* the one we described in Section 2) on $\mathbb{M} \times \mathbb{M}$,

and denoted by \succ :

$$(\underline{i}, \underline{j}) \succ (\underline{L}, \underline{m}) \Leftrightarrow (\underline{j}, \mathbf{w}(\underline{j})) \succ (\underline{m}, \mathbf{w}(\underline{m})) \quad \text{or} \quad \underline{j} = \underline{m} \text{ and } \underline{i} > \underline{L}, \quad \forall \underline{i}, \underline{j}, \underline{L}, \underline{m} \in \mathbb{M}.$$

We shall construct some new simple \mathcal{W} -modules with locally finite actions of elements W_i, L_j for sufficiently large $i, j \in \mathbb{Z}$. For any $d \in \mathbb{N}$, we write

$$\mathcal{W}_d = \sum_{i \in \mathbb{N}} (\mathbb{C}W_{i-d} \oplus \mathbb{C}L_i) \oplus \mathbb{C}C_W.$$

Letting V be a simple \mathcal{W}_d -module, then we have the induced \mathcal{W} -module

$$\text{Ind}(V) = \mathcal{U}(\mathcal{W}) \otimes_{\mathcal{U}(\mathcal{W}_d)} V.$$

Moreover, we always assume that the actions of W_0 and C_W on V are scalars h_W and c_W respectively. Now we give a description of the following results for the W -algebra $W(2, 2)$.

Theorem 5.1. *Assume that $d \in \mathbb{N}$ and V is a simple \mathcal{W}_d -module. If there exists $t \in \mathbb{N}$ such that*

- (a) $\begin{cases} \text{the action of } W_t \text{ on } V \text{ is injective,} & \text{if } t \neq 0, \\ 2h_W + \frac{n^2-1}{12}c_W \neq 0, \quad \forall n \in \mathbb{Z} \setminus \{0\}, & \text{if } t = 0, \end{cases}$
- (b) $W_i v = L_j v = 0$ for all $i > t$ and $j > t + d$,

then we have

- (1) $\text{Ind}(V)$ is a simple \mathcal{W} -module;
- (2) the actions of W_i, L_j on $\text{Ind}(V)$ for all $i > t$ and $j > t + d$ are locally nilpotent.

Denote by $\mathcal{W}^{(x,y)}$ the subalgebra generated by W_i, L_j with $i \geq x$ and $j \geq y$. Subsequently, we characterize simple modules over the W -algebra $W(2, 2)$.

Theorem 5.2. *Suppose that S is a simple \mathcal{W} -module with $2h_W + \frac{n^2-1}{12}c_W \neq 0$ for all $n \in \mathbb{Z} \setminus \{0\}$. Then the following conditions are equivalent:*

- (1) *There exists $t \in \mathbb{Z}$ such that the actions of $W_i, L_i, i \geq t$ on S are locally finite.*
- (2) *There exists $t \in \mathbb{Z}$ such that the actions of $W_i, L_i, i \geq t$ on S are locally nilpotent.*
- (3) *There exist $x, y \in \mathbb{Z}$ such that S is a locally finite $\mathcal{W}^{(x,y)}$ -module.*
- (4) *There exist $x, y \in \mathbb{Z}$ such that S is a locally nilpotent $\mathcal{W}^{(x,y)}$ -module.*
- (5) *There exists $d \in \mathbb{N}$ and a simple \mathcal{W}_d -module V satisfying the conditions in Theorem 5.1 such that $S \cong \text{Ind}(V)$.*

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